ANNEX D

Links between age-efficiency and age-price profiles

This annex spells out, at some detail, the links between the age-efficiency profile and the age-price profile in the non-geometric case. A distinction is made between age-efficiency and age-price profiles for individual assets and for cohorts of assets.

We first recall the optimum condition (56) which says that a cost-minimising producer will use capital goods of different age such that their relative productive efficiency equals the relative rentals for these assets. This is supposed to hold for the cohort as a whole as well as for individual assets. Let h_n and $f_n{}^t$ be the cohort's age-efficiency function and user cost, respectively so that $h_n = f_n{}^t/f_0{}^t$ holds and let $g_n(T)$ and $c_n{}^t(T)$ stand for an individual asset's age-efficiency function and user cost so that $g_n(T) = c_n{}^t(T) / c_0{}^t(T)$ holds. The variables for individual assets have been indexed with T to signal their dependence on a service life T that will in general vary between individual assets.

The first task is to verify the form of a cohort age-price function, given a cohort's age-efficiency function. We do so by combining the asset market equilibrium condition (asset prices equal discounted values of expected incomes generated by the asset) with the definition of the cohort's age-price function ψ_n. As earlier in the text, $P_n{}^{tB}$ stands for the price of an n-period old asset at the beginning of period t.

$$\Psi_n = P_n{}^{tB}/P_0{}^{tB}$$

$$= \frac{f_n^t(1+r_{(tB)})^{-1} + f_{n+1}^{t+1}(1+r_{(tB)})^{-2} + f_{n+2}^{t+1}(1+r_{(tB)})^{-3} + \ldots}{f_n(1+r_{(tB)})^{-1} + f_{n+1}^{t+1}(1+r_{(tB)})^{-2} + f_{n+2}^{t+1}(1+r_{(tB)})^{-3} + \ldots}$$

$$= \frac{f_n^t(1+r_{(tB)})^{-1} + f_{n+1}^t(1+i_{tB})(1+r_{(tB)})^{-2} + f_{n+2}^t(1+i_{tB})^2(1+r_{(tB)})^{-3} + \ldots}{f_0^t(1+r_{(tB)})^{-1} + f_1^t + (1+i_{tB})(1+r_{(tB)})^{-2} + f_2^{t+2}(1+i_{tB})^2(1+r_{(tB)})^{-3} + \ldots} \quad (80)$$

$$= \frac{f_n^t(1+i^*_{(tB)})(1+r_{(tB)})^{-1} + f_{n+1}^t(1+i^*_{tB})^2(1+r^*_{(tB)})^{-2} + f_{n+2}^t(1+i^*_{tB})^3(1+r^*_{(tB)})^{-3} + \ldots}{f_0^t(1+i^*_{(tB)})(1+r^*_{(tB)})^{-1} + f_1^t + (1+i^*_{tB})^2(1+r^*_{(tB)})^{-2} + f_2^{t+2}(1+i^*_{tB})^3(1+r^*_{(tB)})^{-3} + \ldots}$$

9. In this expression, the rates of return and the rates of rental price changes have been expressed in real terms. The next step consists of invoking the optimum condition $h_n=f_n{}^t/f_0{}^t$:

$$\psi_n = \frac{f_n^t(1+i^*_{(tB)})(1+r^*_{(tB)})^{-1} + f_{n+1}^t(1+i^*_{(tB)})^2(1+r^*_{(tB)})^{-2} + f_{n+2}^t(1+i^*_{(tB)})^3(1+r^*_{(tB)})^{-3} + \ldots}{f_0^t(1+i^*_{(tB)})(1+r^*_{(tB)})^{-1} + f_1^t + (1+i^*_{(tB)})^2(1+r^*_{(tB)})^{-2} + f_2^{t+2}(1+i^*_{(tB)})^3(1+r^*_{(tB)})^{-3} + \ldots} \quad (81)$$

$$= \frac{h_n(1 + i^*_{(tB)})(1 + r^*_{(tB)})^{-1} + h_{n+1}(1 + i^*_{(tB)})^2(1 + r^*_{(tB)})^{-2} + h_{n+2}(1 + i^*_{(tB)})^3(1 + r^*_{(tB)})^{-3} + \ldots}{(1 + i^*_{(tB)})(1 + r^*_{(tB)})^{-1} + h_1(1 + i^*_{(tB)})^2(1 + r^*_{(tB)})^{-2} + h_2(1 + i^*_{(tB)})^3(1 + r^*_{(tB)})^{-3} + \ldots}$$

It is now apparent that, given a cohort age-efficiency profile h_n, and a real rate of return r^* as well as a term for the real holding gains/losses i^*, a consistent age-price function ψ_n can be derived for the cohort. To simplify matters, the expected real holding gains or losses can be set to equal zero so that the above expression reduces to:

$$\psi_n = \frac{(h_n(1 + r^*_{(tB)})^{-1} + h_{n+1}(1 + r^*_{(tB)})^{-2} + h_{n+2}(1 + r^*_{(tB)})^{-3} + \ldots)}{(1 + r^*_{(tB)})^{-1} + h_1(1 + r^*_{(tB)})^{-2} + h_2(1 + r^*_{(tB)})^{-3} + \ldots}$$

$$= \frac{\sum_{s=0}^{T\max - n} h_{n+s}(1 + r^*_{(tB)})^{-(s+1)}}{\sum_{s=0}^{T\max - n} h_s(1 + r^*_{(tB)})^{-(s+1)}} \tag{82}$$

Thus, the price for n-period old assets in a cohort relative to the price of new asset corresponds to the ratio of the discounted efficiency units left in an n-year old asset relative to those left in a new asset. The efficiency profile h_n represents the age-efficiency profile of the cohort as a whole. It takes account of the fact that over the maximum service life of the asset group, T^{max}, individual assets will have different individual service lives and be retired earlier than T^{max}. In Section13.3, the cohort's age-efficiency profile was computed from age-efficiency profiles $g_n(T)$ of individual assets and a probability density function F_T for retirement as:

$$h_n = \sum_{T=n}^{T\max} g_n(T)F_T \tag{83}$$

The second avenue to be explored is the derivation of the cohort's age-efficiency profile from its age-price profile, This time, the starting point is the cohort's age-price function, ψ_n that we take as an average of the age-price functions of individual assets, $\theta_n(T)$. Akin to individual age-efficiency functions introduced above, these individual age-price functions depend on each asset's service life T. Combined with the retirement probability F_T, one gets:

$$\psi_n = \sum_{T=n}^{T\max} \theta_n(T)\, F_T \tag{84}$$

Again, the asset-market equilibrium and optimality condition invoked earlier come into play. The age-efficiency pattern for a cohort of assets is computed as follows:

$$\begin{aligned} h_n = \frac{f_n^t}{f_0^t} &= \frac{P_n^{tB} r_{(tB)} + d_n^t - z_n^t}{P_0^{tB} r_{(tB)} + d_0^t - z_0^t} \\ &= \frac{P_n^{tB} r_{(tB)} + P_n^{tB}\delta_n(1 + i_{(tB)}/2) - P_n^{tB} i_{(tB)}(1 - \delta_n/2)}{P_0^{tB} r_{(tB)} + P_0^{tB}\delta_0(1 + i_{(tB)}/2) - P_0^{tB} i_{(tB)}(1 - \delta_n/2)} \\ &= \frac{P_n^{tB}(r_{(tB)} + \delta_n - i_{(tB)} + \delta_n\, i_{(tB)})}{P_0^{tB}(r_{(tB)} + \delta_0 - i_{(tB)} + \delta_0\, i_{(tB)})} \\ &= \frac{P_n^{tB}(r_{(tB)} - i_{(tB)} + \delta_n(1 + i_{(tB)}))}{P_0^{tB}(r_{(tB)} - i_{(tB)} + \delta_0(1 + i_{(tB)}))} \end{aligned} \tag{85}$$

$$= \frac{P_n^{tB}(r_{(tB)} - i_{(tB)} + \delta_n\,(1 + i_{(tB)})}{P_0^{tB}(r_{(tB)} - i_{(tB)} + \delta_0\,(1 + i_{(tB)})}$$

$$= \frac{P_n^{tB}(r^*_{(tB)} - i^*_{(tB)} + \delta_n\,(1 + i^*_{(tB)})}{P_0^{tB}(r^*_{(tB)} - i^*_{(tB)} + \delta_0\,(1 + i^*_{(tB)})}$$

Here, the age-efficiency profile has been expressed as a function of the real rate of return, the real rate of holding gains or losses and the rate of depreciation. A simplified version – sufficient for most practical applications is the calculation ignoring real holding gains or losses. Then, the age-efficiency profile corresponding to a depreciation profile is:

$$h_n = \frac{(P_n^{tB} r^*_{(tB)} + \delta_n)}{(P_0^{tB} r^*_{(tB)} + \delta_0)} \qquad \Psi_n = \frac{(r^*_{(tB)} + \delta_n)}{(r^*_{(tB)} + \delta_0)} \tag{86}$$

This, however, is not the end of the story. The cohort depreciation rates δ_n and δ_0 are themselves functions of the cohort age-price profile and this needs to be taken into account when a full expression for the cohort age-efficiency profile should be derived. From the definition of depreciation rates one has $\delta_n \equiv 1\text{-}\psi_{n+1}/\ \psi_n$, or when the cohort price profile is fully written out:

$$\begin{aligned}
\delta_n &\equiv 1 - \psi_{n+1} / \psi_n \\
&= 1 - \frac{\sum_{T=n+1}^{T\,max} \theta_{n+1}(T)\,F_T}{\sum_{T=n}^{T\,max} \theta_n(T)\,F_T} \\
&= \frac{\sum_{T=n}^{T\,max} \theta_n(T)\,F_T - \sum_{T=n}^{T\,max} \theta_{n+1}(T)\,F_T}{\sum_{T=n}^{T\,max} \theta_n(T)\,F_T} \\
&= \frac{\sum_{T=n}^{T\,max} \theta_n(T)\,F_T - \sum_{T=n}^{T\,max} \theta_{n+1}(T)\,F_T}{\sum_{T=n}^{T\,max} \theta_n(T)\,F_T} \\
&= \frac{\sum_{T=n}^{T\,max} (\theta_n(T)\,F_T - \theta_{n+1}(T)\,F_T)}{\sum_{T=n}^{T\,max} \theta_n(T)\,F_T}
\end{aligned} \tag{87}$$

The last two lines followed from the fact that the price of an (n+1)-year old asset with a service life of n years has to be zero, so that $\theta_{n+1}(n) = 0$. In the next step, this expression is inserted into the simplified formula for the cohort's age-efficiency profile above:

$$\begin{aligned}
h_n &= \psi_n\,\frac{(r^*_{(tB)} + \delta_n)}{(r^*_{(tB)} + \delta_0)} \\
&= \sum_{T=n}^{T\,max} \theta_n\,(T)\,F_T\,\frac{(r^*_{(tB)} + \delta_n)}{(r^*_{(tB)} + \delta_0)} \\
&= \frac{(r^*_{(tB)}\sum_{T=n}^{T\,max} \theta_n\,(T)\;\;F_T\sum_{T=n}^{T\,max}\;(\theta_n(T)\;\;F_T - \theta_{n+1}(T)\,F_T)}{(r^*_{(tB)} + \delta_0)}
\end{aligned} \tag{88}$$

$$= \frac{(\sum_{T=n}^{T\,max} r^*_{(tB)}\theta_n(T)\;\;F_T + (\theta_n(T)\;\;F_T - \theta_{n+1}(T)\;F_T)}{(r^*_{(tB)} + \theta_n F_0)}$$

$$= \frac{(\sum_{T=n}^{T\,max} r^*_{(tB)}\,P_n^{tB}(T)\;\;F_T + (P_n^{tB}(T)\;\;F_T - P_{n+1}^{tB}(T)\;F_T)}{P_0^{tB}\,(r^*_{(tB)} + \theta_0 F_0)}$$

$$= \sum_{T=n}^{T\,max} \frac{c_n\,(T)\,F_T}{c_0} = \sum_{T=n}^{T\,max} \frac{c_n\,(T)\,/\,c_0\,(T)\,F_T\,c_0\,(T)}{c_0} = \sum_{T=n}^{T\,max} = \frac{g_n\,(T)F_T\,c_0\,(T)}{c_0}$$

These lengthy derivations produce an interesting result It turns out that the cohort's age-efficiency function is a *user-cost*-weighted average of the age-efficiency functions of individual assets' age-efficiency functions[2]. This is needed for consistency with an age-price function for the cohort of the form $\psi_n = \sum_{T=n}^{T\,max} \theta_n\,(T)\,F_T$. If this version is chosen, it will be no more possible to follow the avenue that starts out with information on age-efficiency patterns and consecutively derive age-price functions for a cohort. This is because construction of the cohort age-efficiency function requires knowledge of user costs c_0 as shown above. To obtain c_0, a measure of depreciation is needed, and therefore an age-price profile. If one wants to use the cohort age-efficiency function as the starting point, one is thus obliged to use the approach shown in the first part of this Annex. This leads to a different cohort age-price function[3]. It is not evident which version is to be preferred.

Note another consistency issue that arises when non-geometric age-efficiency and age-price profiles are used in conjunction with endogenously computed rates of return: given an age-price profile, a rate of return is required to derive a consistent age-price profile. However, the rate of return cannot be derived endogenously unless there is information on depreciation, which in turn requires knowledge of the age-price profile. Inversely, when the age-profile is the starting point, the productive stock is required to compute the endogenous rate of return. But the productive stock hinges on the age-efficiency profile whose derivation requires information on the rates of return. In principle, the issue can be resolved through a system of simultaneous equations, provided a solution exists, or through iterative algorithms. In practice, these are tedious ways of implementing capital measures and it appears that the choice boils down to the use of geometric profiles and/or the use of exogenous rates of returns.

Notes

1. Katz (2007) points out that "…some countries in Western Europe have used a life of 50 years, which would yield a depreciation rate of 3.2 %. In contrast, because the United States now uses a 0.91 declining balance rate for residential structures, this corresponds to a geometric depreciation rate of 1.14% for 1-4 unit dwellings and a rate of 1.4 % for 5-or more unit dwellings. In comparison, the United States uses rates that are more than double these geometric depreciation rates for major replacements and for additions and alterations to dwellings."
2. The author is obliged to Brian Sliker (U.S. Bureau of Economic Analysis) who demonstrated this in a comment to an earlier version of the document.
3. In principle, thus, there should be a different notation for the cohort's age price and age-efficiency functions, depending on the direction of derivation. We abstained from adding this notational complication.

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