*Chapter 4*

# Problem solving from a mathematical standpoint

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*Problem solving has always been at the heart of both pure and applied mathematics. This chapter examines the evolution of the concept of what constitutes a problem, and the models and findings concerning problem solving from research into the learning and teaching of mathematics. It then considers the emerging understanding of the role played by metacognition in current conceptions of problem solving in mathematics and its learning.*

## Introduction

Mathematics has been described as a search for regularity in patterns, dimension, quantity, uncertainty, shape and change. As a mathematician abstracts what might be a pattern, he or she forms a conjecture of what might constitute a description of that pattern. This then shifts the focus to testing whether related generalisations may be verified. Such searching is at the heart of problem solving. In the abstracting of information, one is engaged in understanding, selecting strategies and representations, and viewing both mentally and figuratively a multitude of instances and potential cases to form a statement describing the problem and its context. In the verification, one is striving to find a solution (Steen, 1990).

While problem solving is at the heart of mathematics, there is little agreement in the field about what problem solving is, what its boundaries are, or how it is best learned or taught to students in schools or practised in adult life. Gardiner indicates that the word "problem" has a negative connotation suggesting an "unwanted and unresolved tension." On the other hand, "solving" has the notion of "a release of tension" (2008: 995). The juxtaposition of these terms throughout mathematics and mathematics education literature suggests that 1) there must be some known way of releasing the tension; and 2) that this process is something that works and can be taught to those wishing to solve problems. While mathematicians and mathematics educators have suggested various methods and approaches, their success in transferring these practices across varied domains within mathematics is questionable. Gian-Carlo Rota, the famed combinatorialist, wrote: "Why don't we tell the truth? No one has the faintest idea how the process of scientific induction works and by calling it a 'process' we may be already making a dangerous assumption" (1986:263). This caution aside, this chapter examines some of the models and practices proposed by mathematicians, mathematics educators, psychologists and other students of problem solving.

## Mathematicians' views of problem solving

While mathematicians from the time of Plato and Aristotle have made comments on the development of mathematical ideas and the values of deductive and inductive approaches, we begin with the comments of the French mathematician Jacques Hadamard (1945) describing his introspective study of problem solving, as well as the results from a survey he made of his contemporaries.

Hadamard begins by refuting the notion that there is a "mathematical brain" and that those endowed with one have special powers to create in mathematical venues. He notes "we shall see that there is not just one single category of mathematical minds, but several kinds, the differences being important enough to make it doubtful that all such minds correspond to one and the same characteristic of the brain" (1945: 5). Turning from this, Hadamard examines the preoccupation of psychologists with notions of sudden chance discoveries or discoveries as the result of lengthy periods of systematic, logical reasoning. Mathematicians, in a survey conducted by M. Edmond Maillet, a colleague of Hadamard, discount chance discoveries of major relationships, as well as ideas springing forth from a dream (Allen, 1888). However, they did ascribe discoveries to having worked hard on a problem, putting it aside for a while, only to return again and then be able to make new progress. Among the responses, there was support for the role that emotions play in problem solving, a topic we will return to later.

Hadamard spent a good deal of time analysing Henri Poincaré's comments on his discoveries based on a presentation he made at the Société de Psychologie in Paris in 1937 (Poincaré, 1952). Poincaré noted several instances of working diligently on problems, setting them aside, and embarking on other tasks, only to have the solution appear to him unexpectedly as he was engaged in other activities. Similar reports are attributed to Gauss, Helmholtz and other members of the scientific community. Hadamard contrasts this sudden clarification of prior efforts from the unconsciousness with the claims of sudden initial discoveries of intact generalisations.

As a result of his study, Hadamard proposed a four-stage model for mathematical problem solving. It consists of a period of preparation, a period of incubation, a period of illumination and a period of verification. The first period, that of preparation, consists of the framing of the question through reading, initial work, analysis of partial results and failed attempts, and conscious examination of potential strategies which may further the work. The second period, that of incubation, consists of the acts of the unconscious mind in combining possibilities and even, potentially, ranking them in terms of "beauty and usefulness" (1945: 32). The third period, that of illumination, is perhaps the least understood or developed. However, it is the stage through which the varied possibilities are brought forward to conscious thought in a more organised and interpretable way than had every before been considered or formulated. Once these new possibilities are on the table, the conscious mind can then assist in the "précising" of these new avenues of attack and the fourth period, that of verification, during the establishment of related solutions to the core problem that initiated the work in the first place. Many mathematicians ascribe to similar events in their work toward finding the solution(s) to problems, both working alone and when working with and on the discoveries of others.

George Pólya, the famous Hungarian mathematician and author of *How to Solve It* (1945), took the discussion of problem solving another step forward in proposing a similar four-step method accompanied by a large number of examples. Pólya's contribution was aimed at teachers and students and shifted the consideration of problem solving from strictly pure mathematical or mathematical physics problems to a wider class of mathematics contexts. In it, Pólya popularised the notion of heuristics or specialised methods or strategies that work across classes of problems. Pólya's four-step method was framed by the stages of understanding the problem, devising a plan, carrying out the plan and looking back.

In *How to Solve It* the discussion of mathematical problem solving is presented alongside a series of examples and problem-solving strategies, or heuristics, related to mathematics education settings. Pólya states that "Routine problems, even many routine problems, may be necessary in teaching mathematics but to make the students do no other kind is inexcusable. Teaching the mechanical performance of routine mathematical operations and nothing else is well under the level of the cookbook because kitchen recipes do leave something to the imagination and judgment of the cook but mathematical recipes do not" (1945: 172). Pólya's model is perhaps the best known of the models proposed by mathematicians. Further, its focus, through examples, on a variety of heuristics, both named these approaches and set them up as targets of instruction in teaching "problem solving" to students as a core process in school mathematics.

## What is a problem?

To this point, we have proceeded as if everyone is in agreement about what a problem actually is. Some mathematicians, as in Gian-Carlo Rota's earlier quote, note the extreme difficulty that exists in communicating the differences between real problem solving and working exercises using a specific strategy. This is the difficulty that surfaces in reducing problem solving to textual material and in attempting to assist teachers to come to recognise, internalise and model problem solving in the mathematical sense in their classrooms. This difficulty in delineating what a "problem" is clouds communication with teachers and the public about the difference between practising routine exercises and engaging in real "problem solving".

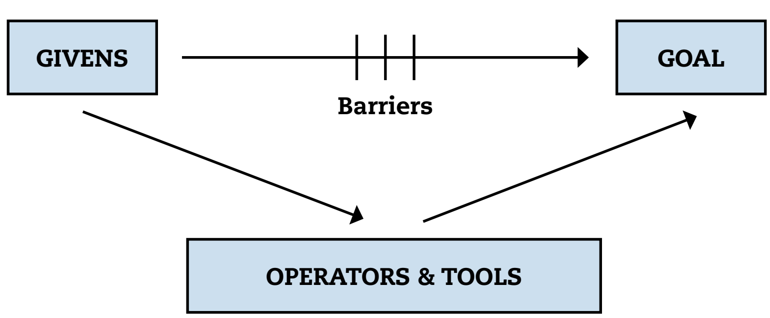
What is unique about a mathematical problem is that it involves a situation, posed in either an abstract or contextual setting, where the individual wrestling with the situation does not immediately know how to proceed or of the existence of an algorithm that will immediately move toward a solution. Further, there is a lingering sense of "unknowingness" about the situation. However, the statement or a representation of the problem situation is within the individual's ken.

Problem solving requires those being asked to participate in searching for a solution to be capable of understanding what the problem itself is. From a mathematical standpoint, this brings up one necessary condition. The problem solver must either know the relevant concepts' definitions or be capable of finding and deciphering them should they not know them. Further, the problem solver needs to understand how to structure mathematical generalisations and representations in such a way as to communicate his or her understanding and the path to a solution or a set of solutions. Further, the individual has to be willing and capable of actually participating in the search for a solution.

This said, let us return to the consideration of what a "problem" actually is. A "problem" exists when an individual has a particular goal, but doesn't know how to achieve it (Duncker, 1945). Problems can range from the task of finding what number to add to 29 to get a sum of 73 for a five-year-old, to resolving the Riemann conjecture for a professional mathematician. In both cases, there is a barrier between the problem solver's current knowledge and the desired result. The presence of this barrier helps conceptually structure the nature and understanding of a problem situation. In either of these cases, there is a lack of a critical connection based on conceptual knowledge or the lack of an appropriate algorithmic approach that stymies the problem solver. In other cases, it may be the lack of knowledge of a theorem (a principle or generalisation), an emotional block or lack of access to tools that creates the barrier between the solver and the solution. As a result of one or more of these impediments, the situation may remain and the problem stays unsolved. Depending on a student's motivation – another source of barriers – the individual may or may not return to try to solve the problem at another time (Funke, 2010).

Figure 4.1 provides a visual representation of the nature of a problem, with the givens and goal state shown with barriers preventing an immediate solution. The alternative path, avoiding the barriers, provides a route to the goal via operators (selections leading to relevant information from heuristic strategies, new representations, expert assistance) and information from tools, such as computer software, calculation devices and measurement instruments (Mayer, 1990; Frensch and Funke, 1995; Mayer and Wittrock, 2006; OECD, 2005).

Figure 4.1 **Problem Situation**




*Source:* Frensch, P.A. and J. Funke (eds.) (1995), Complex Problem Solving: The European Perspective, Erlbaum Associates, Hillsdale, NJ.

The OECD's Programme for International Student Assessment (PISA) structured its definition of problem-solving competency for its 2012 assessment of 15-year-olds' problem-solving capabilities as follows:

> Problem-solving competency is an individual's capacity to engage in cognitive processing to understand and resolve problem situations where a method of solution is not immediately obvious. It includes the willingness to engage with such situations in order to achieve one's potential as a constructive and reflective citizen. (OECD, 2013: 122)

The last sentence in the OECD's conception of problem-solving competency is recognition that competencies are important factors in how individuals interact with and shape situations they encounter in their lives. It reflects that a person competent in solving problems approaches a problem in a reflective and self-monitored, responsible way. This suggests the role that metacognition plays in problem solving competence, a topic that will be discussed later.

## The role of problem solving in the development of mathematics

Problems have played a role in mathematics from the beginning of recorded history. In fact, some of the oldest mathematical artefacts are semblances of modern day mathematical textbooks, with examples of algorithms and comments on their application to the solution of problem-like situations. One of the oldest is the Ahmes, or Rhind, papyrus located at the British Museum. It consists of some 87 mathematical problems written for developing problem-solving competency in students of the period circa 1650 BC (Gillings, 1972). The mathematics education of individuals over successive centuries was guided in large by similar "texts" that provided algorithmic procedures and then practised their application to solving "problems" representative of the use of mathematics in the learners' culture.

However, at the turn of the 20th century, mathematicians like Felix Klein in Germany, John Perry in England and Eliakim Hastings Moore in the United States issued a call for a shift in emphasis, providing a greater focus on developing individual thinking skills and lessening the emphasis on rote memory and practice in the learning of mathematics. Others issued concerns for the development of these process skills to be balanced with some continued practice of more traditional procedural skills (Stanic and Kilpatrick, 1989).

Stanic and Kilpatrick divide the focus on problem solving surfacing in the learning of mathematics into three roles: problem solving as context, problem solving as skill and problem solving as art. The first of these roles sees problem solving as providing contexts that serve as means to other ends. These means include justifying the study of mathematics itself, motivating learners, providing recreation to encourage students, introducing a mathematical concept or skill, or providing practice in a newly encountered algorithm or concept. The second role sees problem solving in the learning of mathematics as an end goal of the act itself. However, this role can sometimes have a dark side in practice, as it leads to a segregation of problems into routine and non-routine problems, with the emphasis in learning often shifting to an increased focus on routine problems. The third role for problem solving is that of problem solving as an art. This interpretation embraces the work of Pólya and other proponents of problem solving who differentiate between active reasoning and plausible reasoning, as Pólya liked to term it (Pólya, 1954). He differentiated active reasoning from the finished reasoning presented in polished, formal deductive proofs. He embraced the use of active reasoning through the inspection of students immersed in a good problem, working their way through towards a solution, working individually or in groups. Further, he urged teachers to assume the roles of posers of problems, of supportive guides providing motivation, of questioners that caused students to reflect on their work, and as co-investigators with their students.

Key to this third conception of problem solving and the related active roles for teachers are two distinctions. The first is between exercises and problems. To Pólya, and most mathematicians and mathematics educators, the key feature of a problem is that one does not have an immediate plan of what to do. This conception of problem solving shows it as an interactive process cycling between the known and the unknown.

The second distinction is between a list of problem-solving strategies and problem-solving itself. Schoenfeld (1992) provides a thoughtful analysis of the multitude of often-contradictory conceptions of problem-solving instructional practices and their implementation in contemporary classrooms. In summarising the research, Begle had the following summary of the state of the field:

> A substantial amount of effort has gone into attempts to find out what strategies students use in attempting to solve mathematical problems…. No clear-cut directions for mathematics education are provided by the findings of these studies. In fact, there are enough indications that problem solving strategies are both problem- and student-specific often enough to suggest that finding one (or few) strategy which should be taught to all (or most) students are far too simplistic. (1979:145)

While pessimistic, Begle's summary was directly on point. Many statements about problem-solving strategies had been reduced to algorithmic recipes for teaching problem solving, often illustrated with examples that were, at best, exercises for the level of students for which the recipes were being touted. Writing a decade later, Dossey and colleagues found little had changed:

> Instruction in mathematics classes is characterized by teachers explaining material, working problems at the board, and having students work mathematics problems on their own - a characterization that has not changed across the eight-year period from 1978 to 1986.
>
> Considering the prevalence of research suggesting that there may be better ways for students to learn mathematics than listening to their teachers and then practicing what they have heard in rote fashion, the rarity of innovative approaches is a matter for true concern. Students need to learn to apply their newly acquired mathematics skills by involvement in investigative situations, and their responses indicate very few activities to engage in such activities. (Dossey et al., 1988:76)

Subsequent retrospective examinations of students' and teachers' reports of learning activities and overall student performance have failed to change this view of classroom instruction and the time devoted to achieving real problem-solving capabilities in students in United States classrooms. Information from other countries suggests that although substantial time is allocated to problem solving, students' capabilities have not yet reached the levels of expectations held by their school programmes (Törner, Schoenfeld and Reiss, 2007).

## Students' problem solving as viewed through PISA

The PISA 2003 special assessment of cross-curricular problem solving (OECD, 2005) presented 15-year-old students in 40 countries with 10 problem situations that together contained a total of 19 tasks. These tasks were spread across three types of problem-solving contexts:

- **Decision making:** Decision-making items required students to choose among alternatives under a system of constraints. The major task was to understand the goal state and the constraints acting on the alternatives, then make a decision and evaluate its effectiveness.
- **System analysis and design:** System analysis and design items required students to identify relationships between parts of a system and/or design a system to express the relationships between parts. The major task was to identify the relevant parts of the system and the relationships among them and their relationship to the goal and then to analyse or design a system that captures the relationships and to check or evaluate that it does.
- **Troubleshooting:** Troubleshooting items required students to diagnose and correct a faulty or underperforming system or mechanism. The major task was to understand the features of the system and their relationship to the malfunction, including causal relationships. The solver needs to diagnose the malfunction, propose a remedy and evaluate the implementation of the remedy.

The assessment was presented in a paper-and-pencil assessment. The tasks ranged from multiple-choice items to items requiring students to construct and communicate their solution to the problem situation presented. The complete set of problems and comparative data are available for downloading via OECD (2005).

The scale resulting from students' performances described four distinct levels of problem solvers. The PISA problem-solving scale was set so that the mean student performance across OECD countries, weighted equally, resided at 500 points with a standard deviation of 100 score points. Students scoring below 405 points were described as Below Level 1: weak or emergent problem solvers. Students in this class failed to understand even the easiest items in the assessment or failed to apply the necessary processes to characterise important features or representations of problems. At best, they could only deal with straightforward tasks or make observations requiring few or no inferences.

Students scoring from 405 points up to 499 points are described as Level 1: basic problem solvers. These students could solve problems based on only one data source containing discrete, well-defined information. They understand the nature of problems and can find and organise information related to the major features of a problem. Then they can transfer from one representation to another if that transformation is not too complex, as well as check a number of well-defined conditions within the problem.

Students scoring from 500 points up to 591 points were classed as Level 2: reasoning, decision-making problem solvers. These students could use analytic reasoning skills and solve problems requiring decision making. They could apply inductive and deductive reasoning, reason about cause and effects, and reason with several information combinations including cases requiring a systematic comparison of all possible cases. They were able to combine and synthesise information from a variety of sources and representations in moving to solutions.

Students scoring 592 points and up were labelled as Level 3: reflective, communicative problem solvers. Students at this level not only analyse problems and make decisions, but they are also capable of thinking about the underlying relationships and how they relate to the solution of the problem. Further, they approach their work in a systematic fashion, constructing their own representations along the way to solving and verifying their solutions. These students are capable of communicating the solution to others in writing and through the use of varied representations. Level 3 problem solvers are capable of co-ordinating a large number of conditions, monitoring variables and temporal conditions, and moving back and forth with changes in interconnected variables. Perhaps the most striking aspect about the work of these students is their organisation and ability to monitor their work with a focus on developing a clear and coherent solution to the problem at hand (OECD, 2005).

While a full accounting of student performance can be found in OECD (2013), suffice it to say that the story of the distribution of problem-solving capabilities is in the variation in performance. About half of the students in the OECD PISA 2003 assessment countries and partner economies score at or above Level 2. Seventy percent or more of the students in Finland, Japan, Korea and Hong Kong, China score at Level 2 or above. Less than 5% of the students in Indonesia and Tunisia were able to meet the same criterion. Further, more than one-third of the students in Japan and Hong Kong, China perform at Level 3. Korea had the highest mean score (550) for individual countries and partner economies, though it was not significantly higher than the scores of the trio of Hong Kong, China (548); Finland (548) and Japan (547).

Large-scale assessments such as PISA provide some gross comparisons of countries' student performance and reflect, to some degree, the emphasis placed on aspects of problem solving in their learning experiences. Most enticing from these results, however, is the fact that the higher one goes in the performance rankings of students in large-scale assessments or smaller studies found in master's or doctoral theses, the more self-organisation and monitoring begin to stand out as criteria differentiating students' performance in problem-solving settings. It is this capability to self-monitor and regulate one's resources in problem solving we turn to next. There were similar findings from the results of the PISA 2012 problem-solving assessment which involved computerdelivered complex problems requiring a greater degree of active student engagement with the problems than the 2003 PISA problem-solving assessment. Further chapters in this work will focus on the results of the 2012 PISA problem-solving study.

## The role of metacognition in mathematical problem solving

A common theme running through studies of students' problem-solving behaviour is the observation that successful problem solvers have control over their progress and rely on reflective pauses to organise their observations, abstract potential patterns, form summary statements and then plan additional data gathering or attempts to develop a communication about their solution (Silver, Branca and Adams, 1980; Schoenfeld, 1985). These plans often parallel the behaviour of experts in acquiring the knowledge they need. This observed behaviour is referred to as metacognition. Bransford, Brown and Cocking (1999) define this behaviour as "the ability to monitor one's current level of understanding and decide when it is not adequate … Adaptive experts are able to approach new situations flexibly and to learn throughout their lifetimes. They not only use what they have learned, they are metacognitive and continually question their current levels of expertise and attempt to move beyond them" (pp. 47-48). A great deal of literature has been developed to indicate the role such metacognitive reflections and monitoring play in overcoming students' initial misconceptions and in developing strong conceptual models that gradually replace those misconceptions with solid habits of mind that form a basis for successful problem solving and mathematical inquiry. However, one should not assume that all successful problem solvers would approach a problem in the same fashion, or that the process of understanding and analysing a problem can be made algorithmic (Schoenfeld, 1985, 1989, 1992).

While the exact link between problem solving and mathematical thinking is unknown, various models have been proposed to account for their mutual support of one another. In general the metacognitive models tend to focus on the following five factors (Schoenfeld, 1992; Lester, Garofalo and Kroll, 1989):

- the knowledge base
- problem-solving strategies
- monitoring and control
- beliefs and affects
- instructional practices.

Based on these factors, proposed by Schoenfeld, or similar models, Schoenfeld and others in mathematics education have used them to examine mathematical thinking and problem solving.

### *The knowledge base*

First and foremost in examining problem solving is the amount and nature of knowledge the solver brings to the situation in which the problem solving is to be focused or the interactions a person has with other individuals or environments. Some researchers also focus on the joint roles of actions and their specific embedding in the environment in which the problem is embedded, arguing for the role of situated cognition (Brown et al., 1989; Lave and Wenger, 1991). A third aspect that Schoenfeld discusses is the role that representations (verbal, figural, graphical, tabular and symbolic) play in establishing one's knowledge base (Schoenfeld, 1992).

Schoenfeld argues that the knowledge base factor breaks down into two areas. "What information relevant to the mathematical situation or problem at hand does he or she possess? And how is that information accessed and used?" (1992: 349). The first area speaks to the facts, concepts, principles, strategies, algorithms, beliefs and attitudes the solver brings to the table. Central in this is the logical differences between facts, definitions, concepts, principles, algorithms and strategies and the differing roles they play in exploring contexts, relating information, ordering hypotheses and verifying assertions on the way to solving a problem. The navigation of this information requires the development of a personal epistemology of mathematical knowledge and the development of categories within it. The second area speaks to the structure of memory (Silver, 1987).

One theory is that the long-term memory is structured as a neural network whose vertices represent major chunks of memory and those edges represent connections between those chunks. Here again epistemology comes into play in considering the types of knowledge. Gilbert Ryle (1949) divided knowledge into “knowing that” and “knowing how”. Cooney and Henderson (1971), looking at mathematics content, divided knowledge into deductive, inductive, classifying and analysing, as a means of assisting students to relate different forms of knowledge and apply it in different situations. In another analysis, Henderson (1967) examined how instruction might link aspects of concepts to aid in understanding or structuring instruction. Others have suggested other instances and models for dividing knowledge into scripts, frames or schemata. All indicate that problem solvers tend to find some organisational form for problem contexts and that these forms help govern their behaviour when they encounter similar experiences in the future.

### *Problem-solving strategies*

As already discussed, there are a number of models that have been developed by mathematicians and mathematics educators to codify approaches to problems and their solutions. Pólya (1945) suggested broad strategies of analogy, auxiliary elements, decomposing and recombining, induction, specialisation, variation, and working backward. However, these have been hard to implement on a general scale, even though they have held up with more mathematically inclined audiences.

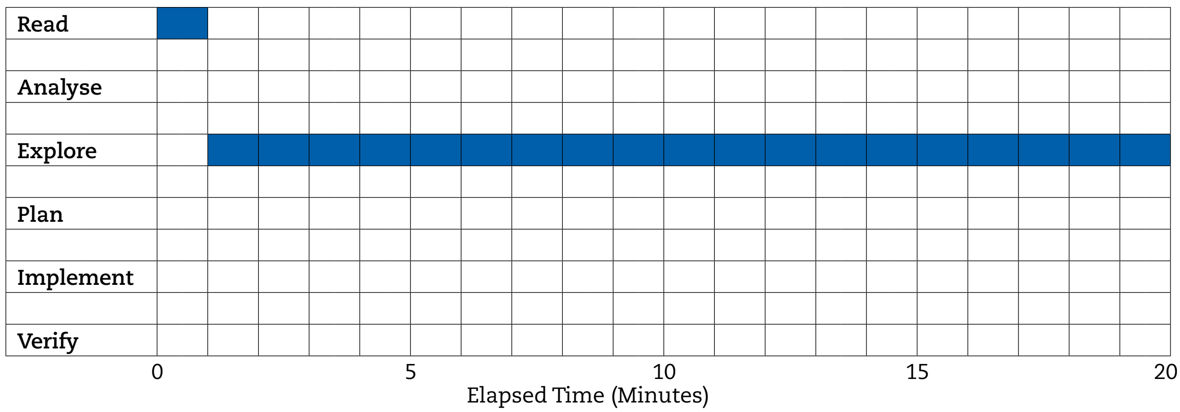
Studies have indicated that, given strategic guidance in structuring learning sequences and focused work with the strategies, combined with enough time, teachers can separate themselves from the “sage on the stage” role to that of “consultant and guide.” (Charles and Lester, 1984). This study and others reflect how difficult it is to implement the power of teachers who can model problem-solving strategies as Pólya envisioned, even with excellent problems and outstanding teachers.

### *Self-regulation, or monitoring and control*

Self-regulation, or monitoring and control, is at the heart of metacognition and, as such, at the heart of mathematical problem solving. Self-regulation is that sixth sense that expert problem solvers have about stopping to check progress or recognising that one needs to change their conceptual framework, representation or tool usage in considering a particular piece of mathematics. The mind is indicating that their present course is not quite as smooth as it should be, or that there is a vital missing link in the explanation being developed. This monitoring is essential in the overall evaluation progress and the selection of the next step in problem solving. Silver, Branca and Adams (1980), Lesh (1985), and Garfalo and Lester (1985) were among the first mathematics educators to see the important role that metacognition played in problem solving, in the form of self-regulation and monitoring.

Perhaps the best illustration of the role of such self-regulation is in the time-effort graphs Schoenfeld (1989) developed from observations of a pair of students, a mathematician and a pair of students’ solving problems after finishing a course in problem solving. The following graphs reflect the activities of problem solvers during intervals of time devoted to attempts to solve problems. Figure 4.2 shows the work of a pair of students attempting to solve a problem.

Figure 4.2 **A pair of students' problem-solving activities over time**

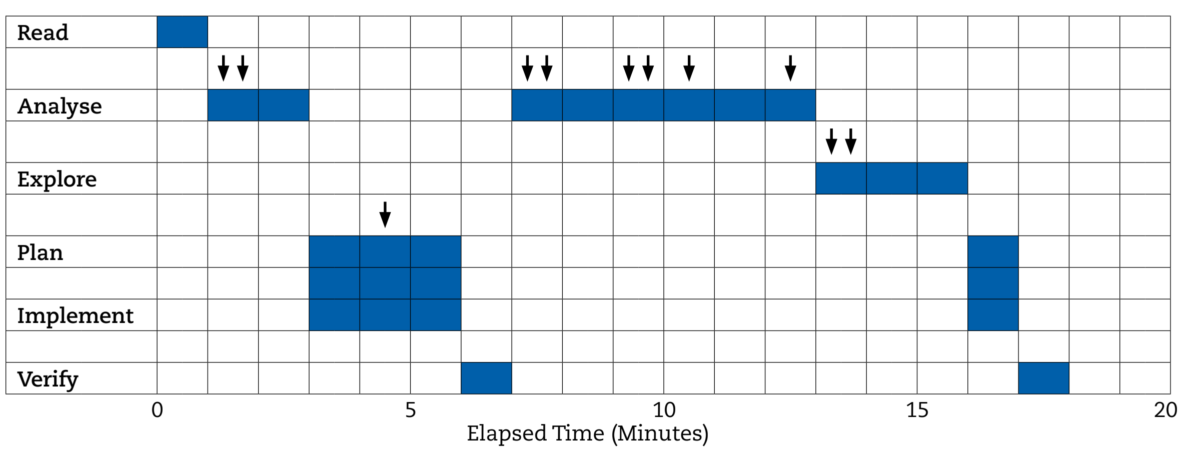



*Source:* Schoenfeld (1989), "Teaching mathematical thinking and problem solving".

An examination of the students' problem-solving activities shows that the pair of students read the problem for a minute or less. They then spent the rest of their time involved in exploring and working on an approach with no signs of any time being allocated to reflective thought focusing on monitoring progress toward a solution. Schoenfeld indicated that this activity-time graph is representative of 60% of the cases of student work he has observed in uninitiated problem solvers.

Figure 4.3 shows the work of a university mathematics professor when confronted with a difficult two-stage problem. This display stands in stark contrast to that of the uninitiated problem-solvers in Figure 4.2.

Figure 4.3 **A mathematician's problem-solving activities over time**




*Source:* Schoenfeld, A. H. (1989), "Teaching mathematical thinking and problem solving" in L.B. Resnick and B.L. Klopfer (eds.), Toward the Thinking Curriculum: Current Cognitive Research, Association for Supervision and Curriculum Development, Washington, DC., pp. 83-103.

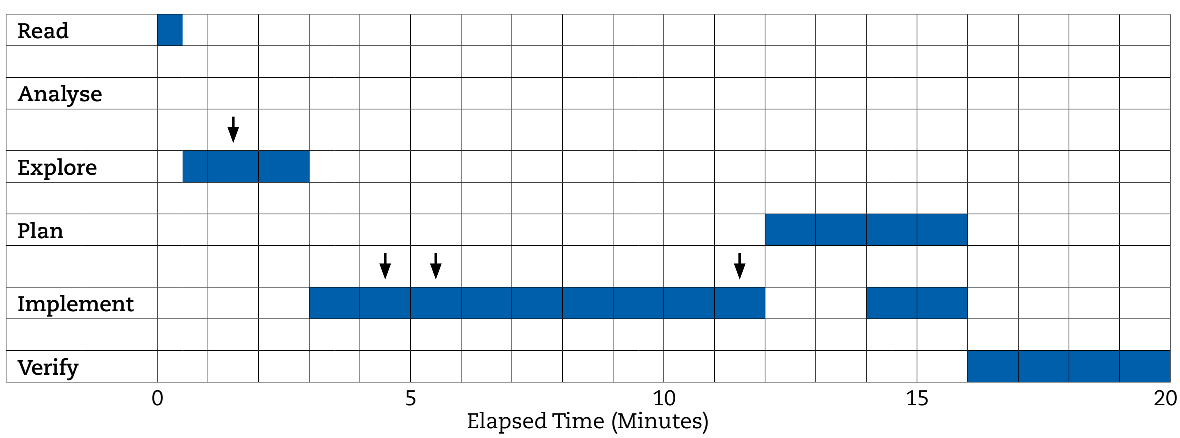
In this graph, the arrows indicate instances where the mathematics professor made an explicit comment about what he was thinking and how it related to the problem solution. Examples were statements such as "Hmm, I don't know exactly where to start here" followed by a shift to two minutes of analysing the problem. This then led to three minutes of cycling through planning and implementation followed by a minute in verification of the first part of the problem. He then returned to the second part of the problem, again making several comments reflective of his metacognitive activities relative to the problem and its solution.

Schoenfeld indicates that he informs students in his class that he will be circulating among the groups while they work on problems. When he visits their group, he is likely to ask one or more of three questions (see Schoenfeld, 1992:356):

- What (exactly) are you doing? (Can you describe it precisely?)
- Why are you doing it? (How does it fit into the solution?)
- How does it help you? (What will you do with the outcome when you obtain it?).

Over the period of a semester, students in the problem-solving class become more adept at answering these questions and even using them as a framework for discussing their solution process among themselves. Figure 4.4 shows an example of an activity-time allocation graph for a pair of students who had completed the problem-solving course while they were working on a problem. Note the change from the activity-time allocation noted in Figure 4.2.

Figure 4.4 **A pair of students' activity-time allocation after a problem-solving course**




*Source: Schoenfeld, A. H. (1989), "Teaching mathematical thinking and problem solving" in L.B. Resnick and B.L. Klopfer (eds.), Toward the Thinking Curriculum: Current Cognitive Research, Association for Supervision and Curriculum Development, Washington, DC., pp. 83-103.*

The allocations here are much more like that of the experienced mathematician in approaching the problem and its solution. Note the understanding (reading), exploring, planning, implementing with monitoring, cycling back to planning and then back to implementing, with a final segment of time in verification activities. The change, documented in times and activities in the transitions from Figures 4.2 to Figure 4.4, shows the type of observations that evolve in students who are immersed in problem-solving contexts with expert feedback. What is clear from the focus on metacognitive behaviour and the thinking aloud interviews with problem solvers is that self-regulation behaviour as part of problem-solving instruction is an important part of developing mathematical thinking skills.

### *Beliefs and attitudes and instructional practice*

Students' approach and avoidance behaviour is shaped to a great extent by their experiences of learning and their environment (McLeod and Adams, 1989). Students' perseverance on a task is often related to their confidence or self-image relative to their ability to do mathematics and to get "correct" answers. Students have a conception of mathematical problems as "trials" that they must endure and that result in a solution if they are able to remember a particular algorithm. When their first attempts fail, they move on to the next thing in the queue on the test, in the book, or in class.

Results of international comparative studies indicate that attitudes and relating mathematical thinking to games and enjoyable activities result in higher performances on assessments including novel problems and other items eliciting problem-solving skills (Chen and Stevenson, 1995; Randel, Stevenson and Witruk, 2000).

Similarly, teachers' attitudes play a large role in how mathematics is conceptualised, structured and presented in the classroom. When teachers are presented with test-mastery materials alongside problem-solving materials, they often opt to focus on the test-mastery material rather than providing experiences with the heuristics that were also the focus of the lesson. Further, there is the underlying foundation of what mathematics itself is. Is it a static set of facts, concepts, generalisations and structures that can be applied to problems or is it a dynamic study of relationships and models that help bring understanding to structures and the links between them in our ever-changing world? The ways in which students' experiences with mathematical thinking occur in the classroom and elsewhere help provide a significant portion of the beliefs and attitudes they form about problems and their solutions (Dossey, 1992; Cooney, 1985; Mevarech and Kramarski, 2014; Thompson, 1985).

### *Summary*

Research on problem solving, along with models considering stages within the process itself, has provided mathematicians and mathematics educators at all levels with a great deal of information and directions for further research. However, much remains to be done to further the development of learners' capabilities to implement heuristics in profitable ways in their problem-solving work in mathematics classes and in their use of mathematics in the world beyond. It is clear that the development of metacognitive actions shows promise towards developing the conditions where the use of heuristics may be profitable. The solving of real problems in classroom learning and in the practice of mathematics, pure or applied, is still a core issue in our world.

## *References*

Allen, G. (1888). "Genius and talent", *Eclectic Magazine of Foreign Literature, Science, and Art*, Vol. 47/4, pp. 448-458.

Begle, E.G. (1979), *Critical Variables in Mathematics Education Findings from a Survey of the Empirical Literature*, Mathematical Association of America, Washington, DC.

Bransford, J.D., A.L. Brown and R.R. Cocking (eds.) (1999), *How People Learn: Brain, Mind, Experience, and School*, National Academy Press, Washington, DC.

Brown, J.S., A. Collins and P. Duguid (1989). "Situated cognition and the culture of learning", *Educational Researcher*, Vol. 18/1, pp. 32-42.

Chen, C. and H.W. Stevenson (1995), "Motivation and mathematics achievement: A comparative study of Asian-American, Caucasian-American, and east Asian high school students", *Child Development*, Vol. 66/4, pp. 1215-1234.

Charles, R.I. and F.K. Lester (1984), "An evaluation of a process-oriented instructional program in mathematical problem solving in Grades 5 and 7", *Journal for Research in Mathematics Education*, Vol. 15/1, pp. 15-34.

Cooney, T.J. (1985), "A beginning teacher's view of problem solving", *Journal for Research in Mathematics Education*, Vol.16/5, pp. 324-336.

Cooney, T.J. and K.B. Henderson (1971), "A model for organizing knowledge", *Educational Theory*, Vol. 21/1, pp. 50-58.

Dossey, J.A. (1992), "The nature of mathematics: Its role and its influence", in D. Grouws (ed.), *Handbook for Research on Mathematics Teaching and Learning*, Macmillan, New York, pp. 39-48.

Dossey, J.A., I.V.S. Mullis, M.M. Lindquist and D.L. Chambers (1988), *The Mathematics Report Card: Are We Measuring Up? Trends and Achievement Based on the 1986 National Assessment*, Educational Testing Service, Princeton, NJ.

Duncker, K. (1945), "On problem solving", *Psychological Monographs*, Vol. 58/5, Whole No. 270.

Frensch, P.A. and J. Funke (eds.) (1995), *Complex Problem Solving: The European Perspective*, Erlbaum Associates, Hillsdale, NJ.

Funke, J. (2010), "Complex problem solving: A case for complex cognition?", *Cognitive Processing*, Vol. 11/2, pp. 133-142, http://dx.doi.org/10.1007/s10339-009-0345-0.

Garfalo, J. and F.K. Lester (1985), "Metacognition, cognitive monitoring, and mathematical performance", *Journal for Research in Mathematics Education*, Vol. 16/3, pp. 163-176.

Gardiner, A. (2008), "The art of problem solving" in T. Gowers, J. Barrow-Green and I. Leader (eds.), *The Princeton Companion to Mathematics*, Princeton University Press, Princeton, NJ, pp. 955-966.

Gillings, R. J. (1972), *Mathematics in the Time of the Pharaohs*, MIT Press, Cambridge, MA.

Hadamard, J. (1945), *The Psychology of Invention in the Mathematical Field*, Dover Publications, New York.

Henderson, K. B. (1967), "A model for teaching mathematical concepts", *The Mathematics Teacher*, Vol .60/6, pp. 573-577.

Lave, J. and E. Wenger (1991), *Situated Learning: Legitimate Peripheral Participation*, Cambridge University Press, New York.

Lesh, R. (1985), "Conceptual analyses of problem-solving performance" in E.A. Silver (ed.), *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives*, Erlbaum Associates, Hillsdale, NJ, pp. 309-330.

Lester, F.K., J. Garofalo and D.L. Kroll (1989), "Self-confidence, interest, beliefs, and metacognition: Key influences on problem-solving behaviour", in D.B. McLeod and V.M. Adams (eds.), *Affect and Mathematical Problem Solving: A New Perspective*, Springer-Verlag, New York, pp. 75-88.

McLeod, D and V. Adams (1989), *Affect and Mathematical Problem Solving: A New Perspective*, Springer-Verlag, New York.

Mayer, R.E. (1990). "Problem solving" in M.W. Eysenck (ed.), *The Blackwell Dictionary of Cognitive Psychology*, Basil Blackwell, Oxford, UK, pp. 284-288.

Mayer, R.E. and M.C. Wittrock (2006), "Problem Solving", in P.A. Alexander and P.H. Winne (eds.), *Handbook of Educational Psychology*, 2nd edition, Erlbaum Associates, Mahwah, NJ:

Mevarech, Z. and B. Kramarski (2014), *Critical Maths for Innovative Societies: The Role of Metacognitive Pedagogies*, Centre for Educational Research and Innovation, OECD Publishing, Paris, http://dx.doi.org/10.1787/9789264223561-en.

OECD (2013), *PISA 2012 Assessment and Analytic Framework: Mathematics, Reading, Science, Problem Solving and Financial Literacy*, OECD Publishing, Paris, http://dx.doi.org/10.1787/9789264190511-en.

OECD (2005), *Problem Solving for Tomorrow's World: First Measures of Cross-Curricular Competencies from PISA 2003*, OECD Publishing, Paris, http://dx.doi.org/10.1787/9789264006430-en.

Poincaré, H. (1952), "Mathematical creation", in B. Ghiselin (ed.), *The Creative Process*, New American Library, New York, pp. 33-42.

Pólya, G. (1945), *How to Solve It: A New Aspect of Mathematical Method*, Princeton University Press, Princeton, NJ.

Polya, G. (1954), *Mathematics and Plausible Reasoning. Vol. I-II,* Princeton University Press, Princeton, NJ.

Randel, B., H.W. Stevenson and E. Witruk (2000), "Attitudes, beliefs, and mathematics achievement of German and Japanese high school students", *International Journal of Behavioral Development*, Vol. 24/2, pp. 190-198.

Rota, G.C. (1986), "More discrete thoughts" in M. Kac, G.C. Rota, and J. T. Schwartz, *Discrete Thoughts: Essays on Mathematics, Science, and Philosophy*, Birkhaüser-Boston, Boston, MA, pp. 263-264.

Ryle, G. (1947), *The Concept of Mind*, Hutchinson, London.

Schoenfeld, A. H. (1992), "Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics" in D. Grouws (ed.), *Handbook for Research on Mathematics Teaching and Learning*, Macmillan, New York, pp. 334-370.

Schoenfeld, A. H. (1989), "Teaching mathematical thinking and problem solving" in L.B. Resnick and B.L. Klopfer (eds.), *Toward the Thinking Curriculum: Current Cognitive Research*, Association for Supervision and Curriculum Development, Washington, DC., pp. 83-103.

Schoenfeld, A.H. (1985), "Metacognitive and epistemological issues in mathematical understanding" in E.A. Silver (ed.), *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives*, Erlbaum Associates, Hillsdale, NJ, pp. 361-379.

Silver, E.A. (1987).,"Foundations of cognitive theory and research for mathematics problem-solving instruction" in A. Schoenfeld (ed.), *Cognitive Science and Mathematics Education*, Erlbaum Associates, Hillsdale, NJ, pp. 33–60.

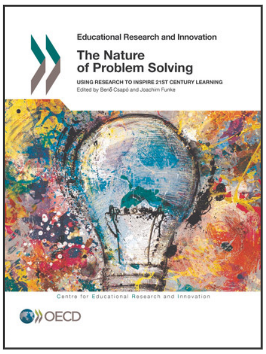
Silver, E.A., N. Branca and V. Adams (1980), "Metacognition: The missing link in problem solving" in R. Karplus (ed.), *Proceedings of the Fourth International Conference for the Psychology of Mathematics Education*, University of California Press, Berkeley, CA, pp. 213-221.

Stanic, G.M.A. and J. Kilpatrick (1989), "Historical perspectives on problem solving in the mathematics curriculum" in R.I. Charles and E.A. Silver (eds.), *Research Agenda for Mathematics Education: The Teaching and Assessing of Problem Solving*, National Council of Teachers of Mathematics, Reston, VA, pp. 1-22.

Steen, L.A. (ed.) (1990), *On the Shoulders of Giants: New Approaches to Numeracy*, National Academy Press, Washington, DC.

Thompson, A. (1985), "Teachers' conceptions of mathematics and the teaching of problem solving" in E.A. Silver (ed.), *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives*, Erlbaum Associates, Hillsdale, NJ, pp. 281-294.

Törner, G., A.H. Schoenfeld and K. M. Reiss (eds.) (2007). "Problem solving around the world: Summing up the state of the art", *ZDM*, Vol/5-6, pp. 353-353.




**From:**

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https://doi.org/10.1787/9789264273955-en

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**Please cite this chapter as:**

Dossey, John A. (2017), “Problem solving from a mathematical standpoint”, in Benő Csapó and Joachim Funke (eds.), *The Nature of Problem Solving: Using Research to Inspire 21st Century Learning*, OECD Publishing, Paris.

DOI: https://doi.org/10.1787/9789264273955-6-en



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